Buckling Analysis of Hydrostatically Loaded Conical Shells by the Collocation Method

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A collocation method is proposed as an alternative to the conventional methods of solving buckling problems. The method is applied to hitherto unsolved buckling problems. Those are elastically supported conical shells of given geometry subjected to hydrostatic pressure. Five possible elastic supports are assumed and the variations of the critical load due to change of the supports' stiffnesses are determined. The suitability of the collocation method for buckling problems analysis is displayed in its ability to satisfy complicated boundary conditions and range differential equations with continuous derivatives.

Introduction

A COMMON method for solving differential equations of a shell is the well-known Galerkin procedure. This method constructs a well-conditioned stability matrix, secures good convergence and enables to cross check at some intermediate stages of the computations. Similar in nature are the Rayleigh-Ritz and the least squares methods. The use of the Galerkin procedure is usually subject to the condition that the assumed displacement functions fulfill the geometric and the natural boundary conditions. However, several hundreds of integrals are involved in the stage of preparation for computer processing. Another problem arises when the critical load (eigenvalue) appears at the boundary conditions. Other methods, such as the known "finite difference" and the "finite element" methods, have their own shortcomings regarding convergence and fulfillment of complicated boundary conditions. A considerable amount of work is now devoted in the application of these methods to more complicated buckling cases.

The "Collocation Method" overcomes the difficulties encountered in the satisfaction of the boundary conditions and in the preparatory work. However, two major problems inherent in the method, namely the determination of collocation points and the ill behavior of the stability matrix, prevented its wide application. These difficulties are treated and overcome in the present work. For more details and references on the various numerical methods see Ref. 1.

The effect of the boundary conditions on the buckling behavior has been a subject of many investigations following the initial ones of Ohira[2] for circular cylindrical shells and Baruch et al.[3] for conical shells. (See also Refs. 1 and 3.)

Very little has been done in analyzing shells with elastic boundary conditions. The importance of solutions which satisfy elastic boundary conditions lies in two major areas: a) in tests, where the actual boundary conditions never behave exactly as the analytically defined boundary conditions. Solutions with elastic boundaries may enable to obtain a better correlation between analysis and tests. b) In the field of shell design, one may find many instances in which several shells are connected together whereby the edge of one shell becomes the elastic boundary for the other shell. The practical cases always include shells' boundaries which should be defined as elastic. A purely analytical solution for the buckling of a semifinite cylindrical shell, axially compressed, with three types of uncoupled elastic supports (k_{N_x}, $k_{N_{x\phi}}$, k_{M_x}), has been presented by Kobayashi.[4]

Although this work deals with a specialized case of the cylindrical shell and does not include the cases of radial elastic constraints, it points out characteristic phenomena in the buckling behavior of the cylindrical shell with elastic supports. Another work by Cohen[5] presents a solution for a stiffened conical shell under hydrostatic pressure with elastic rings at its edges. This type of support, although a practical one, does not represent the pure case of uncoupled elastic supports, and, this restricts one to foresee the influence of each type of displacement on the buckling load. Nevertheless, the results drawn in the work of Cohen[5] are important and check with the results of the present work. Other works which deal with the special case of buckling of various lengths of cylinders with both edges free (this boundary condition is assigned in this work as "SW1") were published by Nachbar and Hoff[6] and Hoff and Soong.[7] Their results are similar in nature to those obtained here for conical shells.

The Collocation Method

Collocation methods have been developed essentially to solve integral equations.[8] It was apparently first applied to differential equations by Slater[9] and independently by Frazer, Jones and Skan[10] and Barta.[11] For an extensive literature review of the collocation method and its application in various fields, see Ref. 1.

The basic procedure for differential equations is as follows: consider an unknown function $y(x)$ which satisfies the differential equation over the interval $[a, b]$

$$L(y) = 0 \quad \text{over} \quad [a, b] \tag{1}$$

and the boundary conditions

$$S(y) = 0 \quad \text{on} \quad S \tag{2}$$

where S defines the boundaries of the interval $[a, b]$. The dependent variable $y(x)$ is approximated by a series expansion $\bar{y}(x)$ containing n undetermined parameters, $a_i (i = 1, \ldots, n)$

$$y(x) \cong \bar{y}(x) = \sum_{i=1}^{n} \bar{y}_i(x) = \sum_{i=1}^{n} a_i p_i(x) \tag{3}$$

The parameters are then determined by applying Eq. (1) at selected points over the interval $[a, b]$ and Eq. (2) at the boundaries. The n algebraic equations which result from collocating at n points (including the boundary equations), are the conditions for solving for the n unknown parameters.

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Generally, the collocation method faces one major difficulty which has to be overcome in order to get reliable solutions.

This problem involves the selection of the collocation points. Comparatively little has been done in selection methods for buckling problems. An equidistant spacing is not generally appropriate in view of the Runge divergence phenomenon[12] for equidistant polynomial interpolation. The present work utilized two criteria for selecting the collocation points.

Consider the linear homogeneous differential equation $L(y)$ with linear homogeneous boundary conditions, $S(y)$

$$L(y) = L_1(y) + \lambda L_2(y) = 0 \qquad (4)$$

$\bar{y}(x)$ will be the approximation to the solving function $y(x)$.

If $\bar{\lambda}$ is the eigenvalue of the approximate system obtained by using the function $\bar{y}(x)$ at the n collocation point and $\lambda^*(x)$ is the eigenvalue computed at any given point, one can define the "local error" of an eigenvalue by

$$\Delta\lambda(x) = \bar{\lambda} - \lambda^*(x) \qquad (5)$$

Equation (4) is satisfied by the approximating function $\bar{y}(x)$ at the collocation points only. At any other point $\bar{y}(x)$ produces an error defined by an error function

$$E(x) = L_1(\bar{y}) + \bar{\lambda} L_2(\bar{y}) \qquad (6)$$

The local eigenvalue $\lambda^*(x)$ is calculated by definition from

$$L_1(\bar{y}) + \lambda^* L_2(\bar{y}) = 0 \qquad (7)$$

By substitution of Eqs. (6) and (7) into Eq. (5) one obtains

$$\Delta\lambda(x) = E(x)/L_2(\bar{y}) \qquad (8)$$

Finally, the relative error of the local eigenvalue is

$$\Delta\lambda(x)/\bar{\lambda} = E(x)/\bar{\lambda} L_2(\bar{y}) \qquad (9)$$

Now, if $E(x)$ approaches zero at every point of the range, the approximate eigenfunction $\bar{y}(x)$ approaches the true eigenfunction $y(x)$ and the approximate eigenvalue $\bar{\lambda}$ approaches the true value of λ. It can be seen therefore, that Eq. (9) can serve as a criterion not only for the eigenvalue but also for the eigenfunction.

The essential problem involves the selection of the best collocation points. Accordingly, one has to set up a criteria for obtaining a "best" approximation of a function $F(x)$, by a set of functions $\phi_i(x)$. For example, one may require that the maximum absolute value of the differences between a function and its approximation

$$D = \left| F(x) - \sum_{i=1}^{N} \phi_i \right|_{\max} \qquad (10)$$

be smaller than, or equal to a known value $D < \varepsilon$. This criterion was selected because of simplicity in operation and is referred to as method A.

An extension of the collocation method is suggested in which one collocates the derivatives of the error function $E(x)$ at the point of maximum error

$$\mathscr{L}_i^{(j)}(\bar{y}) + \bar{\lambda} \mathscr{L}_2^{(j)}(\bar{y}) = 0 \qquad j = 1, \ldots, n \qquad (11)$$

The convergence in method A is erratic. The main advantage of this method is in its simple operation and its straightforward approach. On the other hand the approximating functions in this approach should be chosen carefully. An inappropriate choice of the functions results in divergence, since in this case the number of close collocation points yields a nearly singular stability matrix. A good choice of the approximating functions reduces the collocation points. The number of points in the interval depends also on the interval length and on the precision of the computer.

In conclusion, method A works well when the assumed modes are close enough to the true ones. Otherwise, the criterion is too severe and another method of choosing points should be employed.

The second method (method B) for selection of the collocation points is based on the principle of interpolation by choosing the roots of orthogonal polynomials as interpolation points.[13-15]

It can be shown[1,14] that Gaussian interpolation through the root points of the appropriate orthogonal polynomial, $p_n(x)$, is equivalent from errors summation criterion to interpolation through $m \leq 2n-1$ ordinary range points. This means that by using the n root points, the n degree error function $E(x)$ yields an error over the range equivalent to a $(2n-1)$ degree polynomial. The area under the error function is minimum and observes the least squares principle. Clearly, there is no difference between the error function obtained in interpolation and the error function obtained by the application of the collocation method [Eq. (6)], since both represent an analytic expression of the error within the interval. The difference exists only in the way of obtaining the error function. The approximating function cannot fulfill the differential equation exactly, except for the preselected collocation points. Thus, as in interpolation, the criterion for approximation can be the minimization of the integral of the squares of the error function over the interval, i.e.,

$$E_S = \int_a^b [w(x)E(x)]^2 \, dx \qquad (12)$$

The above should be minimized with respect to the n free coefficient c_{n_i} of the error function polynomial. Thus, it can be shown that its n derivatives yield the following equation

$$\frac{\partial E_S}{\partial c_{n_i}} = 2 \int_a^b w(x)E(x)x^i \, dx \equiv 0, \qquad i = 0, 1, \ldots, n-1 \qquad (13)$$

The second derivative is positive which confirms the minimum property. Clearly, Eq. (13) is satisfied if $E(x)$ is substituted by an orthogonal polynomial $p_n(x)$ with respect to the weight function $w(x)$. The function $E(x)$ can be made equal to $p_n(x)$ by letting the roots of $E(x)$ be equal to the roots of $p_n(x)$. This is achieved by collocating $E(x)$ at the root points of $p_n(x)$. The computations have shown that a unit weight function $w(x) = 1$, yields good results and thus, Gauss points were selected for collocations.

A special importance can be attached to the physical interpretation of this point selection. The differential equations, $L(y)$ [Eq. (4)] consists of generalized forces equation in the y direction. It can be derived by the use of virtual work principle, as it has been done, for example, in Ref. 3. This principle is defined in the following integral

$$\delta U = \int_a^b L(y) \, \delta y \, dx = 0 \qquad (14)$$

Mathematically, an orthogonalization between the generalized forces equation, $L(y)$, and its solution variation, δy, is required. The variation of $y(x)$ approximated by polynomials yields:

$$\delta \bar{y} = \delta a_0 + \delta a_1 x + \cdots + \delta a_{n-1} x^{n-1} \qquad (15)$$

Substituting Eq. (15) in Eq. (14) and letting $w(x) = 1$, one obtains

$$\delta U = \int_a^b E(x)(\delta a_0 + \delta a_1 x + \cdots \delta a_{n-1} x^{n-1}) \, dx = 0 \qquad (16)$$

Equation (16) is essentially identical to Eq. (13) (with $w(x) = 1$) and therefore is satisfied when $E(x)$ is equated to the orthogonal polynomial $p_n(x)$ as already explained. Thus by choosing $w(x) = 1$, and using orthogonal polynomials, the collocation method has been transformed into the principle of virtual work. This identity justifies the selection of $w(x) = 1$.

The advantage of the present approach (method B) of selection points lies in its simplicity, its theoretical background and its efficiency in computer work. The method is convergent, and this quality helps in achieving the solutions efficiently.

As mentioned previously a problem of the collocation method for solution of buckling problems involves the finding of the smallest eigenvalue.

In the process of collocation and satisfaction of the boundary conditions one obtains the following stability algebraic equation

$$[A + \lambda B]\{c\} = 0 \qquad (17)$$

where λ represents the buckling load. The matrix B is usually singular. In spite of this, utilizing the fact that the matrix A, which represents the stiffness of the structure and is always nonsingular, one can calculate the buckling load by the usual "powers" method. In Ref. 1 known buckling loads were com-

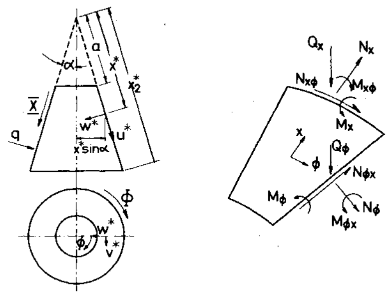

Fig. 1 Notation.

pared with buckling loads calculated by the collocation method. The agreement was found to be excellent.

Buckling Analysis of Conical Shells under Hydrostatic Pressure with Elastic Boundary Conditions

The general expression for the vanishing of virtual work at buckling includes a closed line integral, δU_B, representing virtual work at buckling[16] at the shell's boundaries

$$\delta U_B = \int_0^{2\pi} \Big\{ [(N_x' - \bar{N}_x) a\,\delta u + (N_{x\phi}' - \bar{N}_{x\phi}) a\,\delta v + (\bar{M}_x - M_x')\,\delta w_{,x} + (-\bar{Q}_x + \bar{M}_{x\phi,\phi}/ax \sin\alpha + Q_x') a\,\delta w]\, ax \times \sin\alpha \Big\}_{x=1}^{x=x2} d\phi = 0 \qquad (18)$$

where $\bar{N}_x$, $\bar{N}_{x\phi}$, $\bar{M}_x$, $\bar{M}_{x\phi}$ and $\bar{Q}_x$ are the total external forces and moments acting at the boundaries during buckling and N_x', $N_{x\phi}'$, M_x' and Q_x' are the total internal forces and moments during buckling. The physical components of the displacement (u^*, v^*, w^* which represent the longitudinal, circumferential and normal displacements, respectively) and the geometry of the shell have been normalized with regard to "a" (Fig. 1). By vanishing of each term when multiplied by the arbitrary displacements at the boundaries ($a\delta u$, $a\delta v$, $a\delta w$) or rotation ($\delta w_{,x}$) one obtains the expression for elastic constraints. The "dashed" forces and moments of Eq. (18) consist of the various elastic reactions (Fig. 2) of the boundaries as follows:

$$\bar{N}_x = (-1)^{i+1} k_{Nxi} u^* = (-1)^{i+1} [K_{Nxi} Eh/a(1-v^2)] u^*$$
$$\bar{N}_{x\phi} = (-1)^{i+1} k_{Nx\phi i} v^* = (-1)^{i+1} [K_{Nx\phi i} Eh/2a(1+v)] v^*$$
$$\bar{M}_x = (-1)^{i+1} k_{Mxi} w_{,x} = (-1)^{i+1} [K_{Mxi} Eh^3/12a(1-v^2)] w_{,x}$$
$$\bar{Q}_x = (-1)^{i+1} k_{Qi} w^* = (-1)^{i+1} [K_{Qi} Eh^3/12a^3(1-v^2)] w^*$$
$$\bar{M}_{x\phi} = (-1)^{i+1} k_{Mx\phi i} w^*_{,\phi}/ax \sin\alpha = (-1)^{i+1} [K_{Mx\phi i} Eh^3/12a(1-v^2)] w^*_{,\phi}/ax \sin\alpha \quad \text{at} \quad x = x_i, \quad i = 1,2 \qquad (19)$$

where the k's are the coefficients of the springs per unit length and the K's are their normalized nondimensional counterparts.

By substitution of the forces' expressions in terms of displacements as formulated in Ref. 3 in Eq. (18) the expressions of the various elastic boundary conditions are obtained in the form of displacement functions. In the normal direction of the shell, the expression for the boundary condition is the most complex one, and is coupled with the buckling load. The known boundary conditions SS1–4 and RF1–4 form special cases and are obtained by varying values of the spring coefficients either to zero or to infinity. Other sets of boundary conditions have been defined in the present work by varying one spring at a time equally at both edges, keeping the remaining springs either zero or of infinite value. The commonly defined boundary conditions assume either zero or infinite values of spring stiffness at the boundaries. In the present work intermediate values of spring stiffness are assumed (except those of SW1–4). The relations between the various boundary conditions are described in Fig. 3 where the varying springs are indicated.

Among the infinite spring factors corresponding to a certain intermediate stiffness at the boundaries, one factor is called "nominal." It is defined by assuming the springs to be equal at both edges and by letting the buckling load be equal to the average of the two respective extreme cases (namely zero and infinite spring stiffness). The average value of the buckling load is called the "nominal" value ($SS1W_{nom}$, $SS12_{nom}$...).

Due to the character of the collocation method, the numerical solution scheme of the shell with elastic supports is similar to that of the conventional boundary conditions.

The first two differential equations given in Ref. 3 are solved (term by term) by approximating the displacements by polynomials. Consequently, their approximating functions are:

$$u = \sin t\phi \sum_{i=0}^{m} a_i x^g \quad v = \cos\phi \sum_{i=0}^{m} b_i x^g \quad w = \sin t\phi \sum_{i=0}^{m} c_i x^g \qquad (20)$$

where

$$g = i + \alpha \qquad (21)$$

where α is not necessarily integer, and m is the number of range collocation points plus the number of analytically unfulfilled boundary conditions. Substitution of the displacements yields two linear algebraic equations that yield a_i and b_i as function of c_i. The third stability equation[3,1] and the boundary conditions [Eq. (18)] are then collocated. The matrix scheme of the system is given in Eq. (17). For details see Ref. 1.

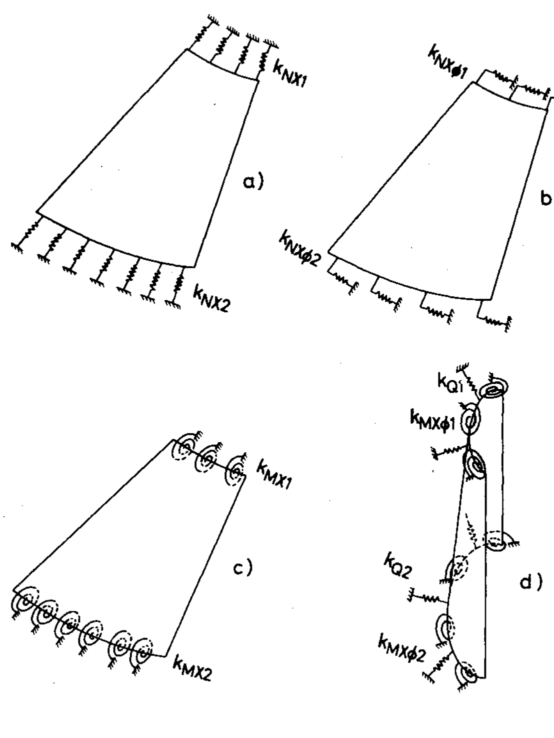

Fig. 2 Elastic springs.

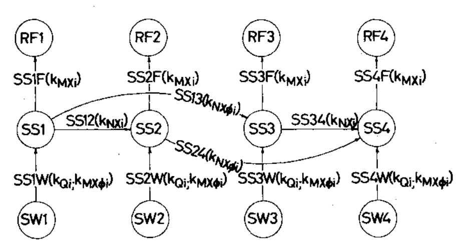

Fig. 3 Scheme of relations between different boundary conditions. (In parantheses are the springs which vary at each stage.)

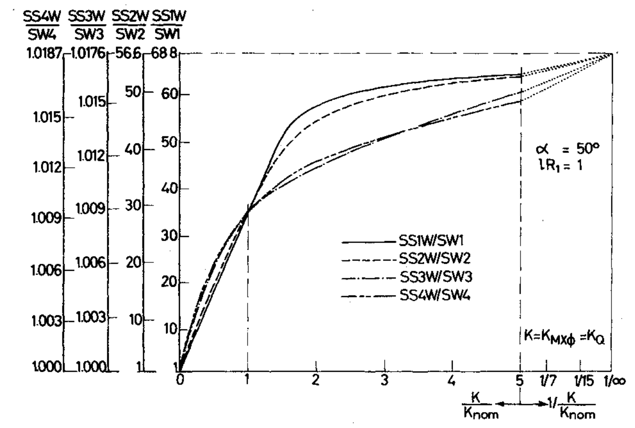

Fig. 4 Normalized variations of buckling loads due to change of springs k_Q and $k_{Mx\phi}$.

Numerical Results and Discussion

Hydrostatic buckling pressure of conical shells with elastic constraints is considered. A symmetry was assumed regarding spring stiffnesses at both edges. Calculations were carried out for a single type of shell geometry. The cone semivertex angle was assumed to be $\alpha = 50°$, length ratio $l/R_1 = 1$, thickness ratio $R_1/h = 100$, and Poisson's ratio $v = 0.3$. In SS1W–SS4W boundary conditions both spring factors K_Q and $K_{Mx\phi}$ were assumed equal. The results obtained for the SW1 to SW4 sets of boundary conditions (i.e., conditions when the normal displacement is free at both edges) allowed to examine the uncoupled influence of each type of restraint at the edges. The condition SW1 presents completely free boundary conditions during buckling. The so called SW2, SW3 and SS1 boundary conditions present the cases where only the longitudinal, the circumferential and the normal displacements are constrained, respectively, while SS2, SS3 and SW4 present the cases where two constraints are applied simultaneously. Comparison of the buckling loads 'p' to those of the free SW1 boundary condition, reveals the influence of each displacement on the shells buckling resistance:

$$\begin{gathered} p_{SW2}/p_{SW1} = 13.51 \qquad p_{SW3}/p_{SW1} = 73.55 \\ p_{SS2}p_{SW1} = 76.55 \qquad p_{SS3}/p_{SW1} = 75.01 \\ p_{SS1}/p_{SW1} = 68.80 \\ p_{SW4}/p_{SW1} = 85.75 \end{gathered} \tag{22}$$

It is interesting to note that the circumferential displacement in the considered geometry of the unstiffened conical shell has the greatest influence at the edges on the shell's resistance and thus can be considered most important there.

Computations were carried out in order to study the variation of the buckling load due to the gradual stiffening of a certain elastic support from zero to infinity. The sensitivity of the boundary condition to a certain elastic support may be defined as the rate of change in buckling load due to small change in the concerned spring stiffness. This sensitivity can be expressed by the slope of the curve at a given point. The nondimensional expressions for the various "sensitivities" S, which are based on graphical measurements are presented as follows:

$$S_{Nx} = \Delta p/\Delta k_{Nx} = (\Delta(p/E)/\Delta K_{Nx})(a/h)(1-v^2)$$
$$S_{Nx\phi} = \Delta p/\Delta k_{Nx\phi} = (\Delta(p/E)/\Delta K_{Nx\phi})2(a/h)(1+v)$$
$$S_{Mx} = \Delta p/(\Delta k_{Mx}h^2) = (\Delta(p/E)/(\Delta K_{Mx}h^2))\,12(a/h)(1-v^2)$$
$$S_{Q;Mx\phi} = \Delta p/\Delta k_Q = (\Delta(p/E)/\Delta K_Q)\,12(1-v^2)(a/h)^3.$$

while

$$K_Q = K_{Mx\phi} \tag{23}$$

The magnitudes of $\Delta(p/E)$ and ΔK are taken from the graphs. In order to compare the sensitivities of the various "classically" defined boundary conditions, calculations were carried out as follows:

$$\text{at } \left.\begin{matrix}SW1\\SW2\\SW3\\SW4\end{matrix}\right\} S_Q = \left\{\begin{matrix}0.188\\0.210\\0.00419\\0.00833\end{matrix}\right. \qquad \text{at } \left.\begin{matrix}SS1\\SS3\end{matrix}\right\} S_{Nx} = \left\{\begin{matrix}1.41\times10^{-5}\\2.45\times10^{-5}\end{matrix}\right.$$

$$\text{at } \left.\begin{matrix}SS1\\SS2\end{matrix}\right\} S_{Nx\phi} = \left\{\begin{matrix}1.56\times10^{-5}\\2.49\times10^{-5}\end{matrix}\right. \qquad \text{at } \left.\begin{matrix}SS1\\SS2\\SS3\\SS4\end{matrix}\right\} S_{Mx} = \left\{\begin{matrix}6.82\times10^{-5}\\8.22\times10^{-5}\\3.99\times10^{-5}\\8.48\times10^{-5}\end{matrix}\right. \tag{24}$$

The above results for the $SW_j\,(j = 1,\ldots,4)$ boundary conditions show again a great sensitivity to the normal displacement at the edges when the shell is not circumferentially restrained (SW1; SW2) and the contrary is true when the shell is circumferentially restrained (SW3; SW4).

Similarly, it can be seen that when the normal displacement w is restrained the axial and circumferential displacements have more or less the same influence. The SS4 boundary condition is the most sensitive to the angular rotation at the edges.

Figures 4 and 5 describe the normalized variations of the buckling load due to various spring stiffnesses. The sensitivity to the elastic supports decreases with the stiffening of the elastic support. Naturally, at infinite stiffness the sensitivity approaches zero. The deflection lines (w vs x) for the SS1W and SS3W boundary conditions are presented in Figs. 6 and 7. Each figure includes three deflection lines for the two extreme cases and for the "nominal" case. The differences in shape reflect their

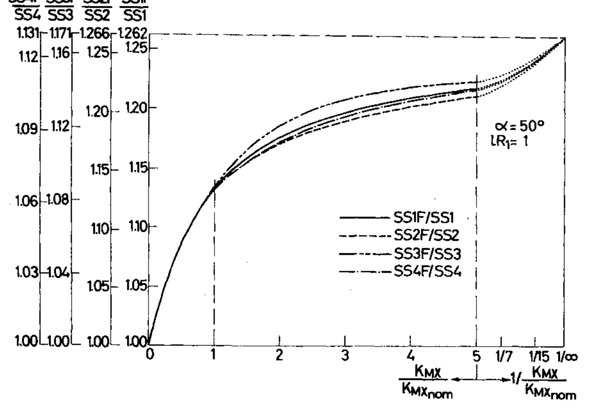

Fig. 5 Normalized variation of buckling loads due to change of springs k_{MX}.

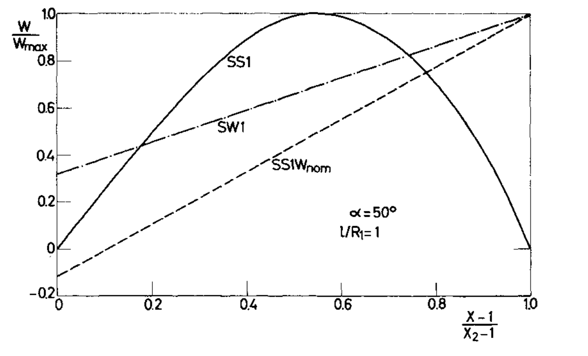

Fig. 6 Buckling modes due to change of springs k_Q and $k_{Mx\phi}$ for SW1, SS1W$_{nom}$ and SS1 boundary conditions.

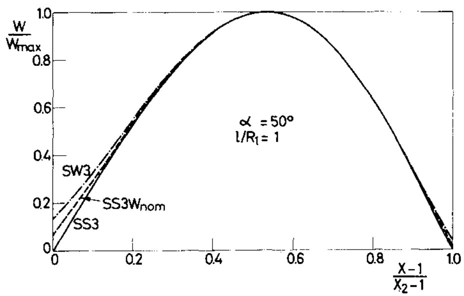

Fig. 7 Buckling modes due to change of springs k_Q and $k_{MX\phi}$ for SW3, SS3W$_{\text{nom}}$ and SS3 boundary conditions.

differences in buckling loads. The deflections at SW1 boundary conditions are almost straight lines with the maximum deflection at the shell's large base. On the other hand the case of SW3 boundary conditions yields similar deflection shapes, with little variation in the buckling loads, for both the extreme cases and the nominal one.

Conclusions

A collocation method has been proposed for solving buckling loads of various systems. This method has proved its efficiency in operation and in accuracy when orthogonal polynomial considerations were used in the selection of the collocation points. Problems with complicated boundary conditions have found their solutions with relative ease.

By assuming elastic supports to an unstiffened conical shell with a given geometry, a certain characteristic variation of the buckling load due to the stiffening of the supports has been detected. The circumferential restraint was found to be of primary importance.

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